Interpretation and limits of sustainability tests in public finance

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Abstract

Public debt is considered sustainable if discounted net repayments are expected to cover the initial debt issuance, i.e. if the government intertemporal budget constraint is expected to hold. With risk averse lenders and an uncertain economic environment, Bohn (1995) stresses that this constraint relies on a stochastic discount factor which depends on lenders' preferences. To get round the difficulty related to the specification of private agents' preferences in empirical analysis, Bohn (1998) suggests to estimate fiscal reaction functions describing how primary surplus reacts to indebtedness. After having solved the econometric issues arising when primary surplus and debt have a very different persistence (with a non-parametric approach) or are both integrated (with parametric tests), we estimate fiscal reaction functions for France and for Greece. The empirical results highlight the remaining limitations and interpretation difficulties that plague these econometric sustainability tests.

Keywords: Intertemporal budget constraint; unit-roots; cointegration; fiscal reaction function; non-parametric tests

Interprétation et limites des tests de soutenabilité des finances publiques

Résumé

La dette publique est dite soutenable lorsque la somme actualisée des remboursements anticipés couvre l'émission initiale de dette, c'est-à-dire lorsque la contrainte budgétaire intertemporelle de l'État est vérifiée. Dans un contexte où les prêteurs sont averses au risque et où l'environnement économique est incertain, Bohn (1995) montre que cette contrainte fait intervenir un facteur d'escompte stochastique qui dépend des préférences des agents. Afin de contourner la difficulté liée à la spécification de ces préférences dans les études empiriques, Bohn (1998) suggère d'estimer des fonctions de réaction fiscale décrivant comment le surplus primaire réagit à l'endettement. Après avoir résolu les difficultés qui se présentent en pratique lorsque le surplus primaire et la dette ont des persistances très différentes (avec une approche non-paramétrique) ou sont tous les deux intégrés (avec une approche paramétrique), nous estimons des fonctions de réaction fiscale pour la France et pour la Grèce. Les résultats mettent en évidence le fait que les tests économétriques de soutenabilité présentent des difficultés d'interprétation et des limites importantes, même lorsqu'ils sont correctement spécifiés.

Mots clés : Contrainte de budget intertemporelle ; racine unitaire ; cointégration ; fonction de réaction fiscale ; tests non-paramétriques

Code JEL : C12, E62, F34, H60, H62
Introduction

Public debt sustainability is a major concern in Europe at least since the outbreak of the Euro Area sovereign debt crisis. Public debt dynamics in countries such as Greece is particularly monitored. But even before the Great Recession, many industrial countries showed persistent deficits and an increasing public debt. Evaluating the sustainability of fiscal policy had naturally come under the spotlight. The Pêbereau (2005) and Champsaur-Cotis (2010) administrative reports illustrate this concern for France.

How to define a sustainable debt? A possible definition is given by the IMF (2002): debt has to “satisf[y] the present value budget constraint without a major correction in the balance of income and expenditure given the costs of financing [the government] faces in the market”. Obviously it is not straightforward to precisely determine what a “major correction” is. Does it refer to a change in the reaction function of government’s revenue and spending to the business cycle or to public debt, or to a sizeable adjustment of public finances, without any change to the usual reaction function? If a government already has, in the past, taken successful measures to curb high indebtedness and faces again the same situation, should it be considered insolvent according to the IMF definition?

In this paper we favor the more usual term of sustainability over solvency. It refers to the ability of a government to pay back its debt with the discounted sum of the primary surpluses generated in the future. As Wyplosz (2007) noticed, the notion of sustainability is essentially forward-looking since it is the future balances that matter. Nevertheless public debt sustainability is often assessed with econometric tests on past data. Boissinot et al. (2004) concluded with standard tests that French public debt was (weakly) sustainable.

Potential behavioral breaks, in the past or between the end of the sample of available data and the near future, represent a first hurdle to interpret these tests’ results. From a logical point of view, the only question they give an answer to is: does the management of public finances as observed in the past justify that investors buy or refuse to buy public debt? Indeed rational investors buy debt securities only under the condition that the discounted repayments by the government cover the initial debt issuance, i.e. when they believe that the government intertemporal budget constraint holds.

Besides the possible behavioral breaks, Bohn (2007) has underscored that usual tests rely on sustainability conditions that are only sufficient. This weakens their interpretation. As long as investors keep on buying public debt securities, the rejection of a sufficient condition of sustainability can therefore be interpreted in two different ways. Either they expect the government to follow in the future a different policy because it can freely adjust its expenditures and receipts. Or they do base their analysis on the past behavior of public finances, but with different lenses than the econometrician, and another sufficient condition for sustainability could justify their buying of public debt.

Following a review of the usual tests and of the sufficient sustainability conditions they lean on, we focus on the specification of the government intertemporal budget constraint. Bohn (1995) has stressed that writing this constraint with the interest
rate on public debt is not always justified. With risk averse lenders and an uncertain economic environment, this constraint relies on a stochastic discount factor depending on lenders’ preferences.

To get round the difficulty related to the specification of private agents’ preferences in empirical analysis, Bohn (1998) suggests to estimate fiscal reaction functions describing how primary surplus reacts to indebtedness. We show that this procedure is valid only under strict technical assumptions and entails econometric difficulties. After having solved the econometric issues arising when primary surplus and debt have a very different persistence (with a non-parametric approach) or are both integrated (with parametric tests), we estimate fiscal reaction functions for France and for Greece in the last part of this study. The empirical results highlight the limitations and interpretation difficulties that plague these econometric sustainability tests. These difficulties are perfectly illustrated in the Greek case as the estimation of the fiscal reaction function on past data indicates that Greek public finances can be considered sustainable in 2007. Similar results for Greece are obtained by Mendoza and Ostry (2008) who also estimate fiscal reaction functions. Conversely, results of non-parametric and parametric tests in the French case are more mitigated but they do not indicate a significantly positive fiscal reaction function over the years 1978-2007. This result is at odds with subsequent developments. Indeed, French public debt has increased markedly after the end of the estimation sample, investors have continued to consider French public indebtedness as sustainable and borrowing costs have remained low, or even very low.

I Literature review on usual indicators and tests

I.1 Frequently used indicators

Public debt sustainability analysis generally starts from the debt accumulation equation: $d_t = \frac{1+i_t}{1+y_t}d_{t-1} - s_t$, $d_t$ being the end of period $t$ stock of debt divided by GDP, $i_t$ the interest rate, $y_t$ the GDP growth rate and $s_t$ the primary balance over GDP.

A simple and widely used indicator is the primary balance stabilizing the debt over GDP ratio. Primary balance (i.e.: balance before interest payments) is the variable under the short term control of the government. The stabilizing primary balance is given by the formula $s_t^* = \left(\frac{1+i_t}{1+y_t} - 1\right)d_t$, or $s_t^* \approx (\bar{i} - \bar{y})d_t$, $\bar{i}$ and $\bar{y}$ being exogenous variables. The intuition is that a country able to stabilize its indebtedness without a major effort can be regarded as safe, a debt reduction requiring only a minimal further improvement of the primary balance. On the contrary, a debt stabilizing primary balance which cannot be reached without a substantial effort signals difficulties to control the evolution of public debt.

In the same spirit, the primary balance required to bring back debt to 60% of GDP (or whatever level considered to be safe) over a given period of time may be computed. Predictable costs to come can also be taken into account, typically the ageing costs (OECD; EC, 2011), or those stemming from a possible bailout of the banking system (Benassy-Quéré, 2012; EC, 2011). The justification for such an

\[1\text{This computation is harder to interpret because it involves episodes with very low probability but far-reaching consequences.}\]
indicator is the following: if the effort to reach a given debt ratio within a reasonable horizon is deemed too large to be credible, then the country’s sustainability can be questioned. The same ideas can be applied to government revenue (or spending) rather than to its surplus in order to answer the following question: by how much must taxes be raised to stabilize the debt to GDP ratio or to balance the budget (Blanchard, 1990)?

All these sustainability analysis are based upon the assumption of exogenous interest rates and GDP growth rates. They abstract from any feedback that fiscal policy may have on these variables (a deficit reduction is not neutral on the growth rate of the economy, either in the short or in the long run, nor on the interest rate). A solution to the first remark is to repeat the analysis with different scenarios for the path of the exogenous variables. If sustainability is accepted (respectively rejected) for a pessimistic (optimistic) scenario, one can be more confident in their diagnosis than with only a central scenario. However conflicting results, depending on the evolution scenarios for the exogenous variables, may also appear. This is the approach chosen by the IMF and the World Bank for their joint Debt Sustainability Assessments, which are studied by Wyplosz (2007). They consist in attaching probabilities to different scenarios for the years ahead and infer a range of possible levels for public debt.

I.2 Economic framework

With $D_t$ the end of period $t$ stock of debt, $r_t$ the interest rate and $S_t$ the primary balance in period $t$, the accounting identity governing the evolution of the stock of debt is: $D_t = D_{t-1}(1 + r_t) - S_t$. Variables can be nominal, real or expressed as a ratio of GDP. The interest rate is thus respectively the nominal rate $i_t$ paid on debt, the real rate or the rate defined by $1 + r_t = (1 + i_t)/(1 + y_t)$, $y_t$ standing for nominal GDP growth. $D_t$ can be written as a function of expected surpluses with a simplifying assumption on future interest rates. The literature often considers $r_t = r > 0$ constant, or $E_t[r_{t+1}] = r > 0$ for example. One obtains recursively:

$$D_t = \sum_{i=1}^{N} \frac{E_t[S_{t+i}]}{(1 + r)^i} + \frac{E_t[D_{t+N}]}{(1 + r)^N}$$ (1)

With rational lenders, the supply of debt meets a demand if the transversality condition (TC ad hoc) holds. It is the case when the expected debt discounted at a rate $r$ converges to 0, meaning that lenders expect that debt will be paid back in full through discounted expected primary surpluses, which is exactly the intertemporal budget constraint (IBC ad hoc):

$$D_t = \sum_{i=1}^{\infty} \frac{E_t[S_{t+i}]}{(1 + r)^i}$$ (IBC ad hoc)

$$\lim_{N \to \infty} \frac{E_t[D_{t+N}]}{(1 + r)^N} = 0$$ (TC ad hoc)

Both preceding constraints are equivalent, and they are valid when debt is sustainable.

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2This choice for the discount rate cannot be justified according to Bohn (1995), and it makes the transversality condition and the intertemporal budget constraint ad hoc. This issue will be addressed in section II.1.
Within this framework it appears that stabilizing the debt to GDP ratio is not a necessary sustainability condition. Debt can be regarded as sustainable even if the ratio of debt over GDP increases whenever the interest rate is larger than the growth rate, which is the case when the economy is dynamically efficient.

The relevant choice of the discount rate will be dealt with later (see II.1). For now we address the interpretation issue of the usual econometric tests: are they based on sufficient and necessary sustainability conditions, or only sufficient ones?

I.3 Usual econometric tests and their limits

One of the first major contributions to the econometric literature on public debt sustainability is Hamilton and Flavin (1986). In this article, the interest rate is constant: it is the ex post real interest rate earned on one-period government bonds over an average year. The authors test the transversality condition \( \lim_{N \to \infty} \mathbb{E}_t[D_t + N] \frac{1}{(1 + r)^N} = 0 \) against the alternative that the limit exists and is strictly positive: \( \lim_{N \to \infty} \mathbb{E}_t[D_t + N] \frac{1}{(1 + r)^N} = A_0 > 0 \). Under the alternative and given past data, economic agents expect part of the debt never to be paid back. In this case the accounting identity (2) can be rewritten:

\[
D_t = \mathbb{E}_t \sum_{i=1}^{\infty} \frac{S_{t+i}}{(1 + r)^i} + A_0(1 + r)^t
\] (2)

If debt follows a process of type (2) and if \( \{S_t\} \) is stationary, it is then equivalent to test the stationarity of \( D_t \) and the nullity of \( A_0 \). Hamilton and Flavin find that \( S_t \) and \( D_t \) are stationary\(^4\) and infer the nullity of \( A_0 \) and the sustainability of the American public debt.

Other authors have chosen an approach based on the variables determining debt’s variation, such as the deficit or public spending and revenue.

Trehan and Walsh (1988) assume that the interest rate is constant and that revenue \( T_t \) and (without interest) spending \( G_t \) are at most I(1). In this case, they show that the transversality condition (TC ad hoc) holds if total deficit (including interests) is stationary. This condition is equivalent to the existence of a cointegrating relationship between \( G_{rt} \) (spending with interests) and \( T_t \) (revenue).

Quintos (1995) further proves that a I(2) debt is compatible with the transversality condition \( \lim_{N \to \infty} \mathbb{E}_t[D_{t+N}] \frac{1}{(1 + r)^N} = 0 \).\(^5\) In this case, debt \( \mathbb{E}_t[D_{t+N}] \) behaves, in probability, like a polynomial in \( N \),\(^6\) but this polynomial is asymptotically dominated by the exponentially-growing discount factor. Bohn (2007) broadens this result and proves

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\(^3\)These authors make substantial efforts to improve debt and deficit data (to subtract from the deficit the interest payments and the seigniorage revenue for instance, or to deal with the gold stock in the United States). The notation \( S_t \) is therefore slightly different in their paper.

\(^4\)ADF test: rejection of the unit root hypothesis at 10 % (but not 5 %) for debt and primary balance.

\(^5\)This is what Quintos (1995) calls weak sustainability. Boissinot et al. (2004) show that French public finances are weakly sustainable over the period 1978-2002 because general government’s expenditures and receipts are bound by a cointegrating relationship \( T_t = \alpha + \beta \cdot G_{rt} + \epsilon_t \) with \( 0 < \beta < 1 \). However, they report that the coefficient \( \beta \) is declining over time and lower than in the U.S. and other European countries.

\(^6\)One can use Bohn’s (2007) proposition 1 to prove \( \mathbb{E}_t[D_{t+N}] = O_p(N^2) \) when \( D_t \sim I(2) \) process.
that sustainability holds when debt is integrated of arbitrarily high order (i.e.; when debt is $I(m)$). Hence, the existence of a cointegrating relationship between revenue $T_t$ and spending (including interests) $G_t^r$ is not a necessary sustainability condition either. If these variables are integrated of orders $m_T$ and $m_G$ respectively, but not cointegrated, the order of integration of debt will be $m$ with $m \leq \max (m_1, m_2) + 1$, ensuring that the transversality condition $\lim_{N \to \infty} \frac{E_t[D_{t+N}]}{(1+r)^N} = 0$ holds.

Procedures testing if debt is integrated of order $m$, against an alternative where its order of integration is strictly larger than $m$, therefore cannot reject the transversality condition. Sustainability holds both under the null and the alternative. In a nutshell, the null hypothesis is a sufficient but not necessary sustainability condition.

Trehan and Walsh (1991) do not restrict debt to be $I(m)$. They show that the transversality condition holds if primary surplus $S_t$ and debt $D_{t-1}$ are linked by a stationary linear combination ($S_t - \alpha D_{t-1} = \epsilon_t \sim I(0)$) and if the quasi-difference of the primary surplus $S_t - \lambda S_{t-1}$ with $\lambda \in [0, 1 + r]$ is $I(0)$ and has a zero mean. It follows:

$$D_t - \lambda D_{t-1} = (S_{t+1} - \lambda S_t) / \alpha - (\epsilon_{t+1} - \lambda \epsilon_t) / \alpha \sim I(0)$$

With $\lambda$ in the interval $[1, 1 + r]$, debt is explosive but sustainable because it is discounted by $(1 + r) > \lambda$. Bohn (2007) notices that debt is not $I(m)$ in this case, whatever the order of integration $m$. This underscores once again that having an $I(m)$ public debt is just a sufficient sustainability condition, but not a necessary one.

Wilcox (1989) has a special place in the econometric literature on sustainability. The variable of interest in his article is real debt discounted at date $(t+i)$ with the realized yield on public debt between date $t$ (reference year) and date $(t+i)$, i.e.: $D_{t+i} (1+r)^i$. It is necessary for the transversality condition to hold that discounted real debt is stationary with its unconditional mean equal to zero. In this case, $\lim_{N \to \infty} E_t[D_{t+N} (1+r)^N] = 0$. Estimating the unconditional mean of the discounted real debt is the purpose of Wilcox’s test.

Wilcox’s framework is not the same as in Bohn (2007) where it is debt in level (nominal or real) which is integrated of order $m$. It can be shown that debt cannot be at the same time $I(m)$ in level and stationary after discounting (see annex A.3). It is however possible to generalize Bohn’s results to a wider class of processes including those analyzed by Bohn and Wilcox (proof in annex A.4):

**Proposition I.1.** Let $f$ be a deterministic and discrete function of time.

1. If $D_{t+n}/f(n) \sim I(m)$, with $m \geq 0$ and $f(n) = o((1+r)^n/n^m)$, then debt verifies the transversality condition (TC ad hoc).

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7 Bohn (2007) notices that Trehan and Walsh (1991) implicitly consider a positive fiscal reaction function: $S_t = \alpha D_{t-1} + \epsilon_t$ with $0 < \alpha$.

8 However, it is not an isolated contribution. The method described by Wilcox (1989) has recently been adapted by Davig (2005) to allow for behavioral breaks in the data generating process of discounted debt.

9 When discounted real debt is not stationary, two cases are possible: either the conditional mean $E_t[D_{t+N} (1+r)^N]$ does not have a limit when $n$ tends to infinity, or it is equal to a random variable and not necessarily to 0.
2. If \( D_{t+n}/f(n) \sim I(0) \) with \( f(n) = O((1+r)^n) \) and \( \mathbb{E}[D_{t+n}/f(n)] = 0 \), then debt verifies the transversality condition (TC ad hoc).

In particular, whatever \( r_0 < r \), \( D_{t+n}/(1+r)^n \sim I(m) \) is a sufficient condition for the ad hoc transversality condition to hold. Furthermore, if \( D_{t+n}/(1+r)^n \sim I(0) \) with mean zero, then the transversality condition also holds: it is the particular case studied by Wilcox (1989). Hamilton and Flavin (1986) also belong to this special case, since equation (2) shows that the conditions of the second case are verified when \( A_0 = 0 \). Finally the first case with a constant function \( f \) corresponds to the entire set of processes considered by Bohn (2007).

All the articles described so far make use of the interest rate on public debt in the transversality condition and, except Wilcox (1989), they make one of the following assumptions to characterize the evolution of this interest rate:

- \( r_t = r > 0 \).
- \( r_t \) is not autocorrelated and \( \mathbb{E}_t [r_{t+1}] = r > 0 \).
- \( r_t \) is a stationary process with mean \( r > 0 \).

Beyond the interpretation issues related to the rejection of only sufficient conditions, the previous econometric tests show another weakness, associated with the choice of a very specific discount factor in the intertemporal budget constraint. We now come to this issue.

II Transversality condition and stochastic discount factor

II.1 Transversality condition with stochastic discount factor

In order to illustrate the choice of the relevant discount factor, we follow Bohn (1995) and consider a simplified endowment economy. At each date, a representative agent receives a random endowment \( Y_t \) that he cannot store. A fixed proportion \( g \) of this endowment is consumed by the government. This public consumption is financed by a tax and the issuance of government bonds to be repaid at a later date. In this decentralized economy, the Euler equation determines the government bond yield ensuring that households effectively consume a fraction \( (1-g) \) of their endowment at each date. Using this Euler equation, one can price any security contingent on a specific state of nature being realized. We denote by \( s_t \) the different states of nature, by \( h_t = (s_t, h_{t-1}) \) the history at date \( t \) and by \( \pi(s_t|h_{t-1}) \) the probability of state \( s_t \) being realized conditional on history \( h_{t-1} \) at previous date.

Private agents maximize their intertemporal utility function:

\[
\sum_{i=0}^{+\infty} \mathbb{E}_t \left[ \sum_{h_{t+i}} \beta^i \pi(h_{t+i}|h_t) U[C_{t+i}(h_{t+i})] \right] = \mathbb{E}_t \left[ \sum_{i=0}^{+\infty} \beta^i U[C_{t+i}] \right]
\]
Given the absence of arbitrage at the optimum, one can determine the price (i.e.: how many units of consumption goods) private agents are willing to invest at time \( t \) in exchange for one additional unit in the history \( h_{t+j} \) at time \( t+j \):

\[
q(h_{t+j}|h_t) = \beta^j \pi(h_{t+j}|h_t) \frac{U''[C_{t+j}(h_{t+j})]}{U''[C_t(h_t)]}
\]

The yield of a government bond issued at time \( t \) and offering a total return \((1 + r_t(j))^j\) in every state of nature at time \( t + j \) is given by the following formula:

\[
\frac{1}{(1 + r_t(j))^j} = \sum_{h_{t+j}} \beta^j \pi(h_{t+j}|h_t) \frac{U''[C_{t+j}(h_{t+j})]}{U''[C_t(h_t)]} = \mathbb{E}_t \left[ \beta^j \frac{U''[C_{t+j}]}{U''[C_t]} \right]
\]

In the same way, one can also price government debt at time \( t \) with payoff \( D_{t+n}(h_{t+n}) \) at time \( t + n \) for every history \( h_{t+n} \), given history \( h_t \):

\[
\sum_{h_{t+n}} \beta^n \pi(h_{t+n}|h_t) \frac{U''[C_{t+n}(h_{t+n})]}{U''[C_t(h_t)]} D_{t+n}(h_{t+n}) = \mathbb{E}_t \left[ \beta^n \frac{U''[C_{t+n}]}{U''[C_t]} D_{t+n} \right]
\]

The relevant transversality condition in a stochastic environment becomes:

\[
\lim_{n \to +\infty} \mathbb{E}_t \left[ \beta^n \frac{U''[C_{t+n}]}{U''[C_t]} D_{t+n} \right] = 0 \quad \text{(TC)}
\]

A strictly positive limit implies that private agents could have a higher intertemporal utility by consuming more at time \( t \) and lending less to the government. The country would then refinance itself indefinitely without ever fully repaying the principal as in a Ponzi scheme. Thus, rational investors would not be willing to hold such assets. By contradiction, this shows that the transversality condition (TC) always holds in equilibrium.

This condition is different from the usual transversality condition that is often used in the literature:

\[
\mathbb{E}_t \left[ \beta^n \frac{U''[C_{t+n}]}{U''[C_t]} D_{t+n} \right] = \mathbb{E}_t \left[ \beta^n \frac{U''[C_{t+n}]}{U''[C_t]} \right] \mathbb{E}_t [D_{t+n}] + \text{Cov}_t \left[ \beta^n \frac{U''[C_{t+n}]}{U''[C_t]} , D_{t+n} \right] = \frac{\mathbb{E}_t [D_{t+n}]}{(1 + r_t(n))^n} + \mathbb{E}_t \left[ \beta^n \frac{U''[C_{t+n}]}{U''[C_t]} , D_{t+n} \right]
\]

Depending on the sign of the covariance term, it could be easier or more difficult for the usual transversality condition to hold. In the case of a deterministic economy or with risk-neutral private agents (with \( U'[C_t] = \frac{C_t^{1+\epsilon}}{1+\epsilon} \), \( \epsilon = 0 \Rightarrow U' (C_{t+n}) = U' (C_t) = 1 \), the transversality condition with uncertainty (TC) actually boils down to the usual (TC \textit{ad hoc}). With risk-averse agents in a stochastic economy, both conditions will not in general coincide as it would require \( D_{t+n} \) and \( \beta^n \frac{U''[C_{t+n}]}{U''[C_t]} \) to be uncorrelated. Of course, it is unlikely to have a zero correlation between these two variables because (marginal utility of) consumption certainly depends on the
budgetary and fiscal stance of the government. Bohn (1995) provides an example in which the debt to GDP ratio remains constant. In this example, the ad hoc transversality condition can be rejected whereas the relevant one in a stochastic setting is always satisfied.

In a stochastic setting, the intertemporal budget constraint becomes:

\[ D_t = \sum_{n \geq 0} \left\{ \mathbb{E}_t \left[ \frac{T_{t+n} - G_{t+n}}{(1 + r_t(n))^n} \right] + \text{Cov}_t \left( \beta^n \frac{U'[C_{t+n}]}{U'[C_t]}, T_{t+n} - G_{t+n} \right) \right\} \quad \text{(IBC)} \]

II.2 About the example presented by Bohn (1995)

We consider the endowment economy defined previously and examine the case where the government issues debt so that the debt/GDP ratio measured at the end of each period and in every state of nature is constant:

\[ \frac{D_t(h_t)}{Y_t(h_t)} = d. \]

Taxes are adjusted so that public spending always represents a constant proportion \( g \) of the endowment \( Y_t \). Equilibrium in the goods’ market therefore implies that agents’ consumption is a constant proportion of the endowment at each period: \( C_t = (1 - g)Y_t \). This fiscal policy will be shown to be sustainable according to the transversality condition in a stochastic setting but not always with the usual one.

We use a CRRA instantaneous utility function with risk aversion denoted \( \epsilon \):

\[ U[C_t] = \frac{C_t^{1-\epsilon}}{1-\epsilon}. \]

The evolution of the endowment is supposed to be log-normal:

\[ Y_t = 1 + y_t \quad \text{with} \log(1 + y_t) \sim \mathcal{N}(\mu, \sigma^2). \]

Finally, we assume that the agents’ intertemporal utility is finite given the properties of this process. Therefore the general term of this positive-term series converges to 0 as \( t \) approaches infinity.

\[ \lim_{n \to +\infty} \mathbb{E}_t \left[ \beta^n U[C_{t+n}] \right] = \frac{(1 - g)^{1-\epsilon}}{1-\epsilon} \lim_{n \to +\infty} \mathbb{E}_t \left[ \beta^n Y_{t+n}^{1-\epsilon} \right] = 0 \]

Given our assumptions, the transversality condition in a stochastic setting holds. Indeed,

\[ \lim_{n \to +\infty} \mathbb{E}_t \left[ \beta^n \frac{U'[C_{t+n}]}{U'[C_t]} D_{t+n} \right] = \frac{d(1 - g)^{-\epsilon}}{U'[C_t]} \lim_{n \to +\infty} \mathbb{E}_t \left[ \beta^n Y_{t+n}^{1-\epsilon} \right] = 0 \]

We now write the risk-free rate on a loan between date \( t \) and date \( t + n \):

\[ \frac{1}{(1 + r_t(n))^n} = \mathbb{E}_t \left[ \beta^n \frac{U'[C_{t+n}]}{U'[C_t]} \right] = \mathbb{E}_t \left[ \beta^n \prod_{i=1}^n (1 + y_{t+i})^{-\epsilon} \right] \]

\(^{10}\) A CRRA utility function is used in this example for the sake of simplicity. It is a well-known fact that this model does not allow to reproduce the pattern (i.e. the low values) of the risk-free interest rate with reasonable values of time preference, consumption volatility and risk aversion. But this example only aims at illustrating the differences between the stochastic discount factor and the riskless discount factor in a simple setting. Research in the joint modeling of economic fluctuations and asset returns is still active. We will circumvent the difficulty associated with the relevant specification of the stochastic discount factor in the empirical section of this paper.
We can then compare the riskless rate with the expectation of the debt variable at time $t+n$ in order to see whether or not the usual transversality condition holds:  

$$\frac{1}{(1 + r_t(n))^n} = \beta^n \exp \left( -\epsilon \mu n + \frac{\epsilon^2 \sigma^2 n}{2} \right)$$

$$\mathbb{E}_t[D_{t+n}] = dY_t \mathbb{E}_t \left[ \prod_{i=1}^{n} (1 + y_{t+i}) \right] = dY_t \exp \left( \mu n + \frac{\sigma^2 n}{2} \right)$$

$$\lim_{n \to +\infty} \frac{\mathbb{E}_t[D_{t+n}]}{(1 + r_t(n))^n} = 0 \Leftrightarrow (\epsilon - 1) \mu - (1 + \epsilon^2) \frac{\sigma^2}{2} - \log(\beta) > 0$$

For sufficiently high values of the risk aversion parameter $\epsilon$ and of the variance $\sigma^2$, the usual transversality condition could be rejected whereas the relevant one in a stochastic setting would always be satisfied.

### III Empirical results

To get round the difficulty related to the estimation of a general equilibrium model and to the specification of private agents’ preferences, Bohn (1998) suggests to estimate a fiscal reaction function describing how primary surplus reacts to public debt. He shows that a positive link is a sufficient condition for sustainability. The theoretical justification, given here in appendix, for estimating a fiscal reaction function is adapted from the unpublished appendix of Bohn (1998). We complement the original proof with an important precision concerning the control variables and the error term in the fiscal reaction function. The proof is relevant only if these variables are bounded. We provide a counter-example showing that it does not work with weakly stationary control variables and error term. Details are available in the appendix at the end of the paper.

In practice, we will estimate the following regression where $\mu_t$ comprises control variables and a structural error term:

$$\frac{S_t}{Y_{t-1}} = \alpha + \beta \frac{D_{t-1}}{Y_{t-1}} + \mu_t \quad (3)$$

It is slightly different from the one proposed by Bohn (1998), for two reasons. First, we change the date convention: $D_t$ is now the end of period $t$ debt whereas it is beginning of period $t$ debt in Bohn (1998). Second, we divide primary surplus $S_t$ and debt $D_t$ by the GDP of period $t - 1$. In this way, the right-hand-side variable $\frac{D_{t-1}}{Y_{t-1}}$ is

$^{11}$With the log-normal hypothesis made for the endowment variation, it can be inferred that :

$$\log(1 + y_t) \sim \mathcal{N}(\mu, \sigma^2)$$

$$\Rightarrow \log \prod_{i=1}^{n} (1 + y_{t+i})^{-1} \sim \mathcal{N}(-\epsilon \mu n, \epsilon^2 \sigma^2 n)$$

$$\Rightarrow \mathbb{E}_t \left[ \prod_{i=1}^{n} (1 + y_{t+i})^{-1} \right] = \exp \left( -\epsilon \mu n + \frac{\epsilon^2 \sigma^2 n}{2} \right) = \left( \mathbb{E} \left[ (1 + y_t)^{-1} \right] \right)^n$$

$^{12}$In his 1998 paper, Bohn alternatively considers a stationary or a (strictly) bounded process $\mu_t$. In the appendix available on his web-page, he details the proof with a (strictly) bounded process only.
fully predetermined and the regression coefficient $\beta$ is not subject to an asymptotic simultaneity bias. In the appendix, we prove that a significantly positive $\beta$ in this specification ensures sustainability, relying on the additional assumption that the interest rate on public debt is bounded. In his specification, Bohn divides primary surplus and beginning of period $t$ debt by the GDP of period $t$. He estimates the following regression, with his own date convention on debt:

$$\frac{S_t}{Y_t} = \alpha + \beta \frac{D_t}{Y_t} + \mu_t$$

Because GDP of period $t$ depends on the primary surplus of period $t$, his estimation is subject to a simultaneity bias, even if $D_t$ (beginning of period $t$ debt) is predetermined.

This method entails other econometric difficulties when the persistence of the primary surplus is very different from the persistence of debt. We will describe a non-parametric method in order to deal with this econometric issue and apply it to French data. We will also consider Greek data where primary surplus and debt ratios are both integrated series. In this case, a parametric (cointegration) method will be considered. But first of all, we start with a description of the available data.

### III.1 Data description

French national accounts in base 2000 include a financial account for general government from 1978 on.\textsuperscript{13} General government debt can be either defined as financial liabilities or as financial liabilities net of financial assets (gross debt or net debt thereafter). Notice that none of these definitions exactly match general government debt as it is defined in the Maastricht Treaty. In this treaty, debt consists in a subcomponent of general government’s financial liabilities taken at their book value rather than at their market value.\textsuperscript{14} It is also worth noting that debt, in the different definitions we consider, is never netted from non-financial assets such as land, buildings and infrastructures which are considered more difficult to liquidate if the government needs cash to repay creditors.

Moreover, Reinhart and Rogoff (2010) have constructed long time series of public debt for several countries including France. Reinhart and Rogoff’s series for France coincides with general government debt as it is defined in the Maastricht Treaty after 1978. For the period before 1978, their series most likely represents financial liabilities of the central government only.

From an economic point of view, the most relevant variable seems to be (financial) net debt. Indeed, nationalizations and privatizations from the 1980s and 1990s led to movements of the same sign on the asset and liability sides of the general government balance sheet. Even if net debt may contain more measurement errors than gross debt, we will rely on it in the following for these economic reasons.

\textsuperscript{13}Since May 2011, French national accounts are in base 2005 and encompass financial accounts since 1995 only. In the following, we rely on the last available accounts in base 2000.

\textsuperscript{14}General government debt in the sense of the Maastricht Treaty is defined as the sum of total deposits (F2), securities excluding stocks and derivatives (F3 - F34) and credits registered on the liability side (F4) (cf. Bourges 2007). These aggregates are precisely defined in the 1993 System of National Accounts (1993 SNA).
Fiscal data available in the annual national accounts (base 2000) only cover a time span of 30 years (1978-2007). As quarterly national accounts in the same base are also available, it is possible to obtain a richer sample for our subsequent empirical analysis, i.e. net debt and primary surplus (structural or not) on a quarterly basis from 1980Q1 to 2007Q4. Using net borrowing (B9A) as a quarterly indicator and Chow and Lin’s (1971) method, we estimated a quarterly net debt for the general government. Moreover, we obtained a quarterly structural primary surplus, using a standard quadratic interpolation of annual potential GDP providing a quarterly output gap. Details on the construction of the quarterly data set are available in appendix A.5.

Figure 1: French public debt from 1949 to 2007 and structural primary surplus over 1978-2007 (% of GDP).

The evolution of Greek debt, primary and structural primary surpluses to GDP ratios is depicted on Figure 2. Greece has been able to reduce its primary deficit rapidly during the 1990s after its debt ratio had started to increase at the beginning of the 1980s. Afterwards, it maintained a positive primary surplus until 2002, allowing to stabilize the debt ratio around 100% of GDP. Only Maastricht debt is available for Greece in international databases. The structural primary surplus is obtained using the European Commission estimates of the output gap (AMECO database), and a budgetary sensitivity of 0.43 (as computed in European Commission, 2005). Unfortunately, the statistical approach used for France to obtain higher-frequency fiscal variables could not be replicated for Greece due to the lack of long enough quarterly national accounts.
III.2 Estimation of a fiscal reaction function when primary surplus and debt are integrated

When primary surplus and public debt are both integrated time series, the fiscal reaction function is a cointegrating relationship. A finite-sample bias might appear if the evolution of the debt / GDP ratio is correlated to the primary surplus / GDP ratio (cf. III.3). To eliminate this bias, Stock and Watson (1993) recommend to include leads and lags of public debt’s variation in the regression:

$$\frac{S_t}{Y_{t-1}} = \alpha + \beta \frac{D_{t-1}}{Y_{t-1}} + \sum_{i=-n}^{n} \gamma_i \left( \frac{D_{t+i-1}}{Y_{t+i-1}} - \frac{D_{t+i-2}}{Y_{t+i-2}} \right) + \mu_t$$

(4)

III.3 Estimation of a fiscal reaction function when primary surplus and debt are both stationary, the latter being much more persistent than the former

The following regression has to be estimated (cf. supra for date conventions):

$$\frac{S_t}{Y_{t-1}} = \alpha + \beta \frac{D_{t-1}}{Y_{t-1}} + \mu_t$$

Results of this regression are difficult to interpret when the primary surplus/GDP ratio is stationary and the debt/GDP ratio is also stationary but very persistent.\textsuperscript{15}

\textsuperscript{15}We assume from the start that unit-root tests do not allow us to differentiate between a formally integrated and a very persistent series for usual sample sizes. If one is absolutely sure to regress a stationary series on an integrated one, the true value of the $\beta$ coefficient cannot be different from
This is a pure econometric issue, not an economic one.

First of all, suppose that the regressor is formally I(1). The error term $\mu_t$ is most likely correlated with the evolution of the debt/GDP ratio between the end of period $t-1$ and the end of period $t$. Indeed, an increase in primary surplus leads to a decrease in debt, everything else held equal. In such a case, the estimator $\hat{\beta}$ has a non-standard asymptotic distribution and a finite-sample bias. The bias is present even when the regressor is predetermined as it is in our case. Thus, it is not a simultaneity bias. Despite the superconvergence property of the estimator, this finite-sample bias is particularly impeding for samples of standard sizes and can lead us to over-reject $H_0 (\beta = 0)$ using a Student test with usual critical values.

With a finite sample, the same difficulty arises for time series that are not formally integrated but simply very persistent (cf. Mankiw and Shapiro (1986) for an empirical illustration and Banerjee and Dolado (1988) for a theoretical explanation). The sign and the size of this bias depend on the unknown correlation between the error term and the evolution of the debt/GDP ratio.

The existence of this bias casts doubt on the results obtained from panel regressions with numerous countries having a persistent debt/GDP ratio such as in Mendoza and Ostry (2008) and the study by the European Commission (PFR 2011).

A first way to eliminate the finite sample bias would be to add an additional lag of the debt/GDP ratio in the regression (3), which gives:

$$\frac{S_t}{Y_{t-1}} = \alpha + \beta \frac{D_{t-1}}{Y_{t-1}} + \gamma \frac{D_{t-2}}{Y_{t-2}} + \mu_t \quad (5)$$

Even when the debt ratio is integrated, estimators $\hat{\beta}$ and $\hat{\gamma}$ converge in $\sqrt{T}$ to standard normal distributions centered at $\beta$ and $\gamma$. Indeed, regression (5) can be rewritten differently with these coefficients associated with stationary variables. We use the fact that the difference $\frac{D_{t-1}}{Y_{t-1}} - \frac{D_{t-2}}{Y_{t-2}}$ is stationary. It is a direct application of a theorem by Sims, Stock and Watson (1990). Simulations done by Galbraith and al. (1987) show that using this method yields excellent results in the case of regressors that are not formally integrated processes but only very persistent.

This parametric method could solve the aforementioned econometric problems due to heterogeneous persistence, but we will not rely on it in the present context for two reasons. First, it would require to make strong assumptions on the stochastic properties of the debt to GDP ratio (e.g.: bounded process) to comply with the proof given in appendix if $\frac{D_{t-2}}{Y_{t-2}}$ is considered as a control variable. Second, it is not obvious to extend Bohn’s proof to the case of fiscal reaction functions with lags of the debt/GDP ratio and/or additional lags of the surplus/GDP ratio. Computations become extremely cumbersome in this case.\(^{16}\)

\(^{16}\)Notice that this difficulty is never taken into account in the empirical literature estimating fiscal reaction functions. In particular, Mendoza and Ostry (2008) only report estimation results with an AR(1) residual. The fact that a significantly positive reaction of primary surplus to indebtedness implies sustainability in this case is not proven.
III.4 Non-parametric tests

The problem arising from the correlation between primary surplus innovations and future values of debt can also be solved using non-parametric tests, without additional lags in the equation. According to Campbell and Dufour (1997), if \( \frac{S_t}{Y_{t-1}} \) is independent from the past (in particular from \( \frac{D_{t-1}}{Y_{t-1}} \) under the null hypothesis \( \beta = 0 \)) and has a median \( b_0 \), then the finite-sample exact distribution of the sign statistic \( S_g(b_0) = \sum_{t=1}^{n} u[(\frac{S_t}{Y_{t-1}} - b_0)(\frac{D_{t-1}}{Y_{t-1}} - \hat{m}_{t-1})] \) is known, where \( u(z) = 1 \) if \( z \geq 0 \) and \( u(z) = 0 \) if \( z < 0 \), and \( \hat{m}_{t-1} \) is the empirical median of the first \( t-1 \) observations of the debt ratio. Moreover, if the primary surplus ratio has a continuous and symmetric distribution about \( b_0 \), then we also know the exact distribution of the signed rank statistic \( S_R(b_0) = \sum_{t=1}^{n} u[(\frac{S_t}{Y_{t-1}} - b_0)(\frac{D_{t-1}}{Y_{t-1}} - \hat{m}_{t-1})]R_t^+(b_0) \) where \( R_t^+(b_0) \) denotes the rank of \( |\frac{S_t}{Y_{t-1}} - b_0| \) among \( |\frac{S_1}{Y_0} - b_0|, \ldots, |\frac{S_n}{Y_{n-1}} - b_0| \) sorted in ascending order, that is \( R_t^+(b_0) = \sum_{j=1}^{n} u[|\frac{S_j}{Y_{j-1}} - b_0| - |\frac{S_t}{Y_{t-1}} - b_0|] \).

Both tests rely on the comparison of the signs of \( \frac{S_t}{Y_{t-1}} - b_0 \) and \( \frac{D_{t-1}}{Y_{t-1}} - \hat{m}_{t-1} \). If \( \beta \) is positive, both primary surplus and debt will tend to be above or below their median at the same time, meaning that \( (\frac{S_t}{Y_{t-1}} - b_0)(\frac{D_{t-1}}{Y_{t-1}} - \hat{m}_{t-1}) \) will be more frequently positive than negative. In such a case, the sign statistic \( S_g(b_0) \) and the signed rank statistic \( S_R(b_0) \) will be positive and far from 0. In contrast with a negative \( \beta \), \( \frac{S_t}{Y_{t-1}} - b_0 \) and \( \frac{D_{t-1}}{Y_{t-1}} - \hat{m}_{t-1} \) will generally display opposite signs, entailing sign and signed rank statistics near 0.

When the median \( b_0 \) of the primary surplus ratio is unknown, Campbell and Dufour (1997) propose two strategies. The first strategy consists in computing the above statistics with the empirical estimator \( b_0 \) of the median \( b_0 \) on the whole sample. However, finite sample distributions of test statistics are not available in this case. The second strategy consists in three steps: first, an exact confidence interval of level \( \alpha_1 \) for \( b_0 \) is computed; then, test statistics of level \( \alpha_2 \) are computed for each value inside the confidence interval; finally, these statistics are combined with the confidence interval for \( b_0 \) using Bonferroni’s inequality in order to end up with a finite-sample exact non-parametric test at the desired level \( \alpha_1 + \alpha_2 = \alpha \).

These non-parametric tests have several advantages compared with the frequently used parametric tests. No restriction is imposed either on the correlation between innovations of the primary surplus ratio and future values of the debt ratio, or on the nature of the innovations generating primary surplus and debt: they can be heteroscedastic and follow non-normal distributions. These tests also rely on exact finite-sample critical values. Numerical simulations done by Campbell and Dufour (1997) show that these test statistics do not wrongly over-reject the null hypothesis and display a power at least similar to standard \( t \)-tests in finite samples.

However, these non-parametric tests are only valid under the assumption that the primary surplus ratio is not autocorrelated under the null hypothesis. This assumption seems to be more acceptable if we consider the cyclically-adjusted (i.e. structural) primary surplus ratio rather than the non-cyclically adjusted primary surplus ratio. Therefore, we only present results of the non-parametric tests when the dependent variable is the structural primary surplus ratio. Notice that this choice amounts to follow Bohn’s (1998) advice and to control for the cyclical evolution of the primary surplus.
surplus ratio (see III.5). Campbell and Dufour (1997) also suggest a method to take into account autocorrelated innovations by considering two subsamples. The first one only contains observations at even dates and the second one those at odd dates. A test of level $\alpha/2$ on each of them will actually amount to an $\alpha$-level test on the whole sample.

III.5 Empirical results for France

Fiscal reaction functions are only estimated on the 1978-2007 sample so that national accounts data are definitive and output gap estimates are more reliable. Indeed, the output gap appears to be a potentially important variable determining the primary surplus / GDP ratio and is most likely correlated with the debt to GDP ratio. Bohn (1998) also suggests to include the output gap in the regression. But rather than directly estimating the elasticity of primary surplus to output gap because we would need instrumental variables to do it properly, we rely on the elasticity of 0.5 computed for France by Guyon and Sorbe (2009). We use the output gap computed by the European Commission (AMECO database) rather than an HP filter because the estimate of the European Commission relies on a production function and is therefore more structural. Since most revisions of this series seem to be concentrated on the last 3 or 4 years, the output gap series computed until 2007 is considered to be reliable.

Considering usual stationarity tests (ADF, KPSS and ERS, see Table 1a), both gross and net debt ratios seem to be I(1) whereas the cyclically-unadjusted primary surplus ratio seems to be I(0). However, the KPSS and ERS unit-root tests lead to inconclusive results for the structural primary surplus. Since we consider that it is not possible for these tests to distinguish between formally integrated series and stationary but very persistent series with the available data, we will assume first that both primary surplus ratios and the net debt ratios are formally I(1), and then that all these variables are stationary with possibly high persistence.\footnote{Remember that we only consider cases where surplus and debt have the same order of integration. If they have different orders of integration, the true value of $\beta$ can only be 0.}

First considering that all series are I(1), we apply the Stock and Watson (1993) method to estimate the fiscal reaction function for France. Results are presented in Table 2. The coefficient on the debt to GDP ratio is never significant. A Shin (1994) test does not reject the cointegration hypothesis between debt and the primary surpluses.\footnote{KPSS statistics for the residuals of regression (1) to (4) of Table 2 are respectively 0.073, 0.058, 0.102 and 0.104 to be compared with Shin (1994) asymptotic critical value at 10 % of 0.231.}

If we assume now that the cyclically-adjusted primary surplus and net financial debt ratios are stationary but the latter very persistent, we apply non-parametric tests introduced by Campbell and Dufour (1997) to implement the sustainability analysis. Specification (3) is used and right unilateral tests are computed. The significance of the fiscal reaction coefficient is assessed at a level of 5%. Sign and signed-rank statistics are computed using either the empirical median estimate of the structural primary surplus ratio (median-estimate tests) or a confidence interval for this median (bounds tests).

Results are reported in Table 1b. Using the empirical median estimate on the sample, the null hypothesis cannot be rejected: the $p$-value is 0.64% for the sign
<table>
<thead>
<tr>
<th>Variable</th>
<th>Order of integration</th>
<th>Level</th>
<th>1st difference</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>ADF</td>
<td>KPSS</td>
</tr>
<tr>
<td>$S/GDP(-1)$</td>
<td>0</td>
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<td>0.13</td>
</tr>
<tr>
<td>$S$ struct/$GDP(-1)$</td>
<td>0/1</td>
<td>-2.49</td>
<td>0.17</td>
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<tr>
<td>$D_{net}/GDP$</td>
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<td>-1.58</td>
<td>0.64</td>
</tr>
<tr>
<td>$D_{gross}/GDP$</td>
<td>1</td>
<td>1.68</td>
<td>0.67</td>
</tr>
</tbody>
</table>

Table 1a: France 1978-2007. Order of integration of fiscal variables with t-stat. ***(**,*) indicates rejection of the null hypothesis of non-stationarity (ADF,ERS) at a 1% (5%,10%) level and ††(†,†) rejection of the null of stationarity (KPSS) at a 1% (5%, 10%) level.

<table>
<thead>
<tr>
<th>$S$ struct/$GDP(-1)$</th>
<th>Median-Estimate Tests</th>
<th>Bound Tests</th>
</tr>
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<tbody>
<tr>
<td></td>
<td>$Sg$, Interpretation</td>
<td>$SRg$, Interpretation</td>
</tr>
<tr>
<td>Total sample</td>
<td>0.64, $H_0$ not rejected</td>
<td>0.66, $H_0$ not rejected</td>
</tr>
<tr>
<td>Sub-sample A</td>
<td>0.40, $H_0$ not rejected</td>
<td>0.64, $H_0$ not rejected</td>
</tr>
<tr>
<td>Sub-sample B</td>
<td>0.21, $H_0$ not rejected</td>
<td>0.24, $H_0$ not rejected</td>
</tr>
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</table>

Table 1b: France 1978-2007. Under $H_0$, primary surplus ratios and debt ratios are independent. Right unilateral tests are performed on the statistics $Sg$, $SRg$, $SB$ and $SRB$. $p$-values are indicated in the table. $Q_L$ is the smallest value taken by the test statistic on the confidence interval defined for $b_0$. $Q_U$ is the largest value.

Note: For median-estimate tests, relying on the empirical median estimate $b_0$ of the structural primary surplus / GDP(-1) ratio, significance is tested at a 5% level (2.5% for the subsamples). For bounds tests, a 99% confidence interval $J(0.01)$ is first constructed for the median $b_0$ on the whole sample (99.5% on each subsample). $H_0$ is rejected if, for all $b \in J(0.01)$ ($J(0.005)$ for subsamples), the test statistic is above the 4% critical value (2% for subsamples). $H_0$ is not rejected if, for all $b \in J(0.01)$ ($J(0.005)$ for subsamples), the test statistic is less than the 6% critical value (3% for subsamples). It may occur that $Q_L$ is less than the 4% (2%) critical value but that $Q_U$ is above the 6% (3%) critical value. In this case, test results are said to be inconclusive.
<table>
<thead>
<tr>
<th></th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
<th>(4)</th>
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<tr>
<td></td>
<td>$S/GDP(-1)$</td>
<td>$S/GDP(-1)$</td>
<td>$S\text{ struct}/GDP(-1)$</td>
<td>$S\text{ struct}/GDP(-1)$</td>
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<tr>
<td>$D(-1)/GDP(-1)$</td>
<td>0.0196</td>
<td>0.0203</td>
<td>−0.0131</td>
<td>−0.0126</td>
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<td>(0.0143)</td>
<td>(0.0151)</td>
<td>(0.0175)</td>
<td>(0.0165)</td>
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<tr>
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<td>(0.0582)</td>
<td>(0.0698)</td>
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<tr>
<td>$D/GDP - D(-1)/GDP(-1)$</td>
<td>−0.136</td>
<td>−0.145**</td>
<td>−0.110*</td>
<td>−0.132**</td>
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<td>(0.0849)</td>
<td>(0.0765)</td>
<td>(0.0587)</td>
<td>(0.0489)</td>
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<tr>
<td>$D(-1)/GDP(-1) - D(-2)/GDP(-2)$</td>
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<tr>
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<td>(0.0583)</td>
<td>(0.0777)</td>
<td>(0.0636)</td>
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<tr>
<td>$D(-2)/GDP(-2) - D(-3)/GDP(-3)$</td>
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<td>(0.0532)</td>
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<tr>
<td>Constant</td>
<td>−0.624</td>
<td>−0.914**</td>
<td>−0.469</td>
<td>−0.502</td>
</tr>
<tr>
<td></td>
<td>(0.446)</td>
<td>(0.360)</td>
<td>(0.417)</td>
<td>(0.379)</td>
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<td>Observations</td>
<td>25</td>
<td>27</td>
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<tr>
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<td>0.460</td>
<td>0.428</td>
<td>0.005</td>
<td>0.020</td>
</tr>
</tbody>
</table>

Standard errors in parentheses  
* $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$

Table 2: France 1978-2007. OLS estimates with Newey-West standard errors
statistic and 0.66% for the signed-rank statistic. Using a confidence interval for
the median of the structural primary surplus ratio, the null hypothesis is not rejected
either (cf. details under Table 1b). Like Campbell and Ghysels (1995), we then divide
the sample in two parts and apply the same non-parametric tests on each subsample
so that the assumption of non-autocorrelated residuals becomes more credible. Under
the null hypothesis, this assumption means that structural primary surplus ratios are
non-autocorrelated in each subsample. These robustness checks confirm our previous
results, indicating a lack of response of primary surplus to indebtedness.

We then perform the same analysis with quarterly data interpolated from the
annual national accounts, yielding similar results. Table 3a shows stationarity tests
for these fiscal variables, indicating that net debt and both surplus ratios are \( I(1) \)
and cointegrated. Applying once again the Stock and Watson (1993) method leads
to the same conclusion for French debt sustainability: no evidence of a significantly
positive fiscal reaction function, as reported in Table 3b.

As a conclusion for the French case, results of parametric and non-parametric tests
are mitigated but they do not indicate a significantly positive fiscal reaction function
in the past.\(^{19}\) So, to the extent private investors used the fiscal reaction function as a
device to assess sustainability, they probably anticipated a strengthening of the fiscal
reaction even before the start of the financial crisis to justify their buying of French
public debt. However, French public debt continued to increase after the end of the
estimation sample and French borrowing costs remained low, or even very low.

III.6 Empirical results for Greece

Debt and primary surplus ratios may be both considered as \( I(1) \) for this country.
Therefore, the fiscal reaction function is estimated using Stock and Watson (1993)
method. Results are presented in Table 4b. The coefficient on the debt to GDP ratio
is estimated at around 0.1 for the 1978-2007 period, depending on the specification.
It is always significant at a 1% level. This estimate is very high, implying that a 10
points increase in the debt ratio leads to a 1 point increase in the (structural) primary
surplus, thus pointing strongly towards sustainability. A Shin (1994) test does not
reject the cointegration hypothesis between (structural) primary surplus and debt
ratios.\(^{20}\) Notice that Greece also appears as a country with very sustainable public
finances in Mendoza and Ostry (2008) who also estimate fiscal reaction functions.

Of course, this result may seem confusing when one considers the recent eco-
nomic developments in Greece. This should be an important warning for the users
of econometric sustainability tests. Greece is actually unable to finance its public
debt on the market although its past fiscal reaction function points to a sustainable
indebtedness. In fact, investors probably anticipated that Greece would be unable to
apply this fiscal reaction function at higher debt levels. This is exactly the issue that
Bi and Leeper (2012) deal with using a general equilibrium model. Their conclusion

\(^{19}\)The estimation of a fiscal reaction function for Germany is not considered in this paper. Indeed
the huge challenges following the German reunification probably caused a break in the fiscal reaction
function. Furthermore the treatment of the privatizations of eastern companies in the early 1990's
through the Treuhandanstalt should be taken into account.

\(^{20}\)KPSS statistics for the residuals in regressions (1) to (4) of Table 4b are respectively 0.117, 0.121,
0.104 and 0.112, the Shin (1994) asymptotic critical value at 10% being 0.231.
<table>
<thead>
<tr>
<th>Variable</th>
<th>Order of integration</th>
<th>Level</th>
<th>1st difference</th>
</tr>
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<tr>
<td></td>
<td></td>
<td>ADF</td>
<td>KPSS</td>
</tr>
<tr>
<td>$S/GDP(-1)$</td>
<td>1</td>
<td>-2.13</td>
<td>0.20</td>
</tr>
<tr>
<td>$S_{struct}/GDP(-1)$</td>
<td>1</td>
<td>-2.68*</td>
<td>0.18</td>
</tr>
<tr>
<td>$D_{net}/GDP$</td>
<td>1</td>
<td>-1.69</td>
<td>1.12 †† †</td>
</tr>
</tbody>
</table>

Table 3a: France 1980Q1-2007Q4. Order of integration of quarterly fiscal variables with t-stat. ***(**,*) indicates rejection of the null hypothesis of non-stationarity (ADF,ERS) at a 1% (5%,10%) level and ††(†,††) rejection of the null of stationarity (KPSS) at a 1% (5%, 10%) level.

<table>
<thead>
<tr>
<th></th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
<th>(4)</th>
</tr>
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<tr>
<td></td>
<td>$S /GDP(-1)$</td>
<td>$S /GDP(-1)$</td>
<td>$S_{struct} /GDP(-1)$</td>
<td>$S_{struct} /GDP(-1)$</td>
</tr>
<tr>
<td>$D(-1)/GDP(-1)$</td>
<td>0.003</td>
<td>0.003</td>
<td>-0.003</td>
<td>-0.003</td>
</tr>
<tr>
<td></td>
<td>(0.003)</td>
<td>(0.002)</td>
<td>(0.003)</td>
<td>(0.003)</td>
</tr>
<tr>
<td>$D(+1)/GDP(+1) - D/GDP$</td>
<td>0.005</td>
<td>-0.036</td>
<td>(0.079)</td>
<td>(0.0634)</td>
</tr>
<tr>
<td></td>
<td>(0.079)</td>
<td></td>
<td>(0.0634)</td>
<td></td>
</tr>
<tr>
<td>$D/GDP - D(-1)/GDP(-1)$</td>
<td>-0.206***</td>
<td>-0.182**</td>
<td>-0.143**</td>
<td>-0.188</td>
</tr>
<tr>
<td></td>
<td>(0.069)</td>
<td>(0.072)</td>
<td>(0.066)</td>
<td>(0.061)</td>
</tr>
<tr>
<td>$D(-1)/GDP(-1) - D(-2)/GDP(-2)$</td>
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<td>0.100</td>
<td>0.083</td>
<td>0.094</td>
</tr>
<tr>
<td></td>
<td>(0.056)</td>
<td>(0.062)</td>
<td>(0.058)</td>
<td>(0.058)</td>
</tr>
<tr>
<td>$D(-2)/GDP(-2) - D(-3)/GDP(-3)$</td>
<td>-0.010</td>
<td>-0.158</td>
<td>-0.027</td>
<td>-0.027</td>
</tr>
<tr>
<td></td>
<td>(0.064)</td>
<td>(0.074)</td>
<td>(0.054)</td>
<td>(0.070)</td>
</tr>
<tr>
<td>$D(-3)/GDP(-3) - D(-4)/GDP(-4)$</td>
<td>-0.130</td>
<td>-0.003</td>
<td>-0.158</td>
<td>-0.158</td>
</tr>
<tr>
<td></td>
<td>(0.080)</td>
<td></td>
<td>(0.071)</td>
<td></td>
</tr>
<tr>
<td>Constant</td>
<td>-0.008***</td>
<td>-0.008***</td>
<td>-0.004</td>
<td>-0.004</td>
</tr>
<tr>
<td></td>
<td>(0.002)</td>
<td>(0.002)</td>
<td>(0.003)</td>
<td>(0.003)</td>
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</table>

Table 3b: France 1980Q1-2007Q4. OLS estimates with Newey-West standard errors. $D$ denotes the French general government net debt.
<table>
<thead>
<tr>
<th>Variable</th>
<th>Order of integration</th>
<th>Level ADF</th>
<th>Level KPSS</th>
<th>Level ERS</th>
<th>1st difference ADF</th>
<th>1st difference KPSS</th>
<th>1st difference ERS</th>
</tr>
</thead>
<tbody>
<tr>
<td>S/GDP(-1)</td>
<td>1</td>
<td>-1.68</td>
<td>0.31</td>
<td>4.61</td>
<td>-6.05***</td>
<td>0.14</td>
<td>1.80***</td>
</tr>
<tr>
<td>S struct/GDP(-1)</td>
<td>1</td>
<td>-1.79</td>
<td>0.33</td>
<td>4.74</td>
<td>-5.67***</td>
<td>0.14</td>
<td>1.76***</td>
</tr>
<tr>
<td>D/GDP</td>
<td>1</td>
<td>-1.36</td>
<td>0.66††</td>
<td>158.86</td>
<td>-4.71***</td>
<td>0.25</td>
<td>1.96**</td>
</tr>
</tbody>
</table>

Table 4a: Greece 1978-2007. Order of integration of fiscal variables with t-stat. ***(**, * ) indicates rejection of the null hypothesis of non-stationarity (ADF, ERS) at a 1% (5%, 10%) level and †††(††, †) rejection of the null of stationarity (KPSS) at a 1% (5%, 10%) level.

Table 4b: Greece 1978-2007. OLS estimates with Newey-West standard errors.
is that the default risk does not only depend on a fiscal reaction function but also on the fact that the product of taxes cannot grow indefinitely to stabilize debt above a certain threshold due to economic and social constraints (Laffer curve). It is also possible that the 2000’s increase in the share of foreign holders of public debt has changed the fiscal reaction function or the reaction of the market to new developments in public finance. Given that our analysis uses the most accurate data as of 2012 for the period 1978-2007, it is very likely that the same analysis with 2007 data would have pointed even more strongly towards sustainability. Yet the uncovering of the misleading fiscal figures in 2010 certainly undermined the investors’ confidence in the Greek government, their statistics and their ability to react according to the estimated fiscal reaction function.
Conclusion

This paper has recalled the weaknesses associated with the first generation of sustainability tests in public finance. Some of them can be overcome using Bohn’s suggestion to estimate fiscal reaction functions linking primary surplus and public debt. A positive link is a sufficient sustainability condition, under the strong condition that the control variables and the error term are bounded. In practice, since this method also entails econometric difficulties when primary surplus and debt have a very different persistence or are both integrated, we have put forward parametric and non-parametric methods in order to deal with these issues.

But even the second generation of sustainability tests has strong empirical limitations, as we have shown using French and Greek national accounts over the last 30 years. Because Greece generated an enormous increase of its primary surplus during the 1990s, it appears to fulfill this sufficient sustainability condition in 2007. This result is clearly at odds with recent economic developments in this country. As for France, results of parametric and non-parametric sustainability tests do not indicate a significantly positive fiscal reaction function over the years 1978-2007. Again, this result is at odds with subsequent developments. Indeed, French public debt has increased markedly after the end of the estimation sample, investors have continued to consider French public indebtedness as sustainable and borrowing costs have remained low, or even very low.

Our results highlight the limits of econometric sustainability tests. Even if they are correctly specified, they only give an answer to the following question: is it rational for an investor, using only the past reaction of primary surplus to debt, to lend money to a government? In fact, fiscal reaction functions can evolve strongly according to the circumstances and the sustainability of a country’s public finance depends above all on the quality and strength of its institutions.
A Appendix

A.1 Sufficient sustainability condition based on the fiscal reaction function

This appendix gives a theoretical justification for estimating fiscal reaction functions. It is adapted from Bohn’s (1998) unpublished appendix. Compared to Bohn’s original proof, date conventions have been changed: $D_t$ is now the end of period $t$ debt whereas it is the beginning of period $t$ debt in Bohn (1998) and the interest rate on a loan made in period $t$ is $R_t$ instead of $R_{t-1}$. We also divide primary surplus $S_t$ and debt $D_t$ by the GDP of period $(t-1)$ $Y_{t-1}$ (see III for an econometric justification, related to the estimation of the fiscal reaction function). Finally, the public debt accumulation equation becomes: $D_t = (D_{t-1} - S_t)(1 + R_t)$. This accounting equation means that debt at the end of period $t$ already includes the interest burden that is due for period $(t+1)$. In period $t$, households lend $D_{t-1} - S_t$ to the government and will be repaid $(D_{t-1} - S_t)(1 + R_t)$ in period $(t+1)$. As a consequence, the equation linking the households’ stochastic discount factor and the risk-free interest rate on public debt is $E_t [u_{t+1} : (1 + R_{t+1})] = 1$.

The debt accumulation equation can be divided by the GDP of period $t$:

$$d_t = \frac{D_t}{Y_t} = \left( \frac{D_{t-1} - S_t}{Y_{t-1}} \right) \cdot (1 + R_t) \frac{Y_{t-1}}{Y_t} = \left( \frac{D_{t-1} - S_t}{Y_{t-1}} \right) \cdot x_t$$

Suppose that the fiscal reaction function is of the following form, with $0 < \rho < 1$:

$$s_t = \frac{S_t}{Y_{t-1}} = \rho \frac{D_{t-1}}{Y_{t-1}} + \mu_t$$

Like in Bohn (1998), $\mu_t$ comprises control variables and a structural error term.

Primary surplus can then be taken out of the equation governing the evolution of public debt. By successive iterations, we get:

$$d_{t+n} = \left( \prod_{j=1}^{n} x_{t+j} \right) (1 - \rho)^n d_t - \sum_{i=1}^{n} \left( \prod_{j=i}^{n} x_{t+j} \right) (1 - \rho)^{n-i} \mu_{t+i}$$

We can note that this equation implicitly assumes that there is no default, which underlies the only additional technical assumption that we need compared to Bohn (1998): the interest rate $R_t$ is bounded, i.e. $0 < m \leq 1 + R_t \leq M$. This implies, given the positivity of $u_{t+i,1}$, that: $mu_{t+i,1} \leq u_{t+i,1}(1 + R_{t+i}) \leq Mu_{t+i,1} \Rightarrow \frac{1}{M} \leq E_t [u_{t+i,1}] \leq \frac{1}{m}$.

We deduce in the next steps:

$$E_t [u_{t,n} \cdot D_{t+n} \cdot Y_t] = E_t \left[ u_{t,n} \cdot \prod_{j=1}^{n} (1 + y_{t+j}) \cdot d_{t+n} \right]$$

$$= (1 - \rho)^n \cdot E_t \left[ u_{t,n} \cdot \prod_{j=1}^{n} (1 + y_{t+j}) \cdot \prod_{j=1}^{n} x_{t+j} \right] \cdot d_t$$

$$- \sum_{i=1}^{n} (1 - \rho)^{n-i} \cdot E_t \left[ u_{t,n} \cdot \prod_{j=1}^{n} (1 + y_{t+j}) \cdot \prod_{j=1}^{n} x_{t+j} \cdot \mu_{t+i} \right]$$
\[
(1 - \rho)^n \cdot \mathbb{E}_t \left[ u_{t,n} \cdot \prod_{j=1}^{n} (1 + R_{t+j}) \right] \cdot d_t
\]

\[- \sum_{i=1}^{n} (1 - \rho)^{n-i} \cdot \mathbb{E}_t \left[ u_{t,n} \cdot \prod_{j=i}^{n} (1 + R_{t+j}) \cdot \prod_{j=1}^{i-1} (1 + y_{t+j}) \cdot \mu_{t+i} \right] \]

\[= (1 - \rho)^n \cdot \mathbb{E}_t \left[ u_{t,1} \prod_{j=1}^{n-1} u_{t+j,1} \cdot (1 + R_{t+j}) (1 + R_{t+n}) \right] \cdot d_t \]

\[= (1 - \rho)^n \cdot \mathbb{E}_t \left[ u_{t,1} \prod_{j=1}^{n-1} u_{t+j,1} \cdot (1 + R_{t+j}) (1 + R_{t+n}) \cdot \mu_{t+i} \right] \]

We then use the law of iterated expectations and give an upper bound to the absolute value of \( \mathbb{E}_t[u_{t,n} \cdot D_{t+n}] \). Indeed, all the terms in the integrands except \( \mu_t \) are positive and we assume, like Bohn (1998), that the process \( \mu_t \) is bounded, meaning that there exists a \( M_0 \) such that the realization of \( \mu_t \) in every state of nature and at each date lies between \(-M_0 \) and \( M_0 \). Therefore:

\[
\left| \frac{\mathbb{E}_t[u_{t,n} \cdot D_{t+n}]}{Y_t} \right| \leq (1 - \rho)^n \cdot \frac{d_t \cdot M}{m} \]

\[+ \sum_{i=1}^{n} (1 - \rho)^{n-i} \cdot \mathbb{E}_t \left[ u_{t,i} \cdot \prod_{j=1}^{i-1} (1 + y_{t+j}) \cdot u_{t+i,1} \cdot (1 + R_{t+i}) \cdot |\mu_{t+i}| \right] M \]

\[\leq (1 - \rho)^n \cdot \frac{d_t \cdot M}{m} + MM_0 \sum_{i=1}^{n} (1 - \rho)^{n-i} \cdot \mathbb{E}_t \left[ u_{t,i} \cdot \prod_{j=1}^{i-1} (1 + y_{t+j}) \cdot u_{t+i,1} \cdot (1 + R_{t+i}) \right] \]

\[\leq (1 - \rho)^n \cdot \frac{d_t \cdot M}{m} + MM_0 \sum_{i=1}^{n} (1 - \rho)^{n-i} \cdot \mathbb{E}_t \left[ u_{t,i-1} \cdot u_{t+i-1,1} \cdot \prod_{j=1}^{i-1} (1 + y_{t+j}) \right] \]

\[\leq (1 - \rho)^n \cdot \frac{d_t \cdot M}{m} + \frac{M}{m} MM_0 \sum_{i=1}^{n} (1 - \rho)^{n-i} \cdot \mathbb{E}_t \left[ u_{t,i-1} \cdot \prod_{j=1}^{i-1} (1 + y_{t+j}) \right] \]

By assumption, also made by Bohn (1998), the discounted value of future revenues

\[\sum_{k=1}^{n} Y_t \cdot \mathbb{E}_t \left[ u_{t,k} \cdot \prod_{j=1}^{k} (1 + y_{t+j}) \right] \]

is finite, so that \( \lim_{k \to +\infty} \mathbb{E}_t \left[ u_{t,k} \cdot \prod_{j=1}^{k} (1 + y_{t+j}) \right] = 0 \).

It is then straightforward to show that:

\[\lim_{n \to +\infty} \frac{\mathbb{E}_t[u_{t,n} \cdot D_{t+n}]}{Y_t} = 0 \]

This means that the existence of the postulated fiscal reaction function is sufficient for the transversality condition to hold, whatever the exact form of the stochastic discount factor. Recall that this result is valid under some technical assumptions: the control variables and the error term are bounded, the discounted value of future revenues is finite and the interest rate is bounded.
A.2 Stationarity of the error term and the control variables is not sufficient

It is important to bear in mind that stationarity of the control variables and the error term in the fiscal reaction function together with the positivity of the coefficient on debt is not a sufficient sustainability condition (i.e.: it does not imply the transversality condition (TC)). However, if the control variables and the error term are bounded and if there is a positive $\beta$ in the fiscal reaction function, then (TC) holds. Weaker conditions on the control variables could be found, but neither boundedness in probability (which is implied by weak stationarity), nor weak stationarity, nor even strong stationarity are sufficient. An example with a weakly stationary process is given below, for which having a positive $\beta$ in the fiscal reaction function does not imply (TC). Another example with a strongly stationary process follows.

A.2.1 An example with weak stationarity

Let $u_k \sim iid U([0,1])$. We define $\mu_k$ and $u_t, k \prod_{j=1}^{k} (1 + y_{t+j}) \equiv X_k$ as follows:

$$
\mu_k = \begin{cases} 
\frac{\sqrt{k^4 - 1}}{\sqrt{k^3 - 1}} & \text{if } u_k < \frac{1}{k^2} \\
\frac{1}{\sqrt{k^3 - 1}} & \text{otherwise}
\end{cases}
$$

$$
X_k = \begin{cases} 
k^2 & \text{if } u_k < \frac{1}{k^2} \\
0 & \text{otherwise}
\end{cases}
$$

It is straightforward to show that $\mu_k$ is weakly stationary: $E[\mu_k] = \frac{\sqrt{k^4 - 1}}{k^3} - \frac{1}{\sqrt{k^3 - 1}} \left( 1 - \frac{1}{k^2} \right) = 0$, $\forall [\mu_k] = \frac{k^2}{k^3} + \frac{1}{\sqrt{k^3 - 1}} \left( 1 - \frac{1}{k^2} \right) = 1$, and all the autocorrelations are null given that $u_k$ is iid. Thus, $\mu_k$ is a weakly stationary process.

Furthermore, $E[X_k] = \frac{k^2}{k^3} \to 0$.

However, $E[X_k \mu_k] = \frac{k^2 \sqrt{k^3 - 1}}{k^3} \to \frac{1}{k^3} = 1$.

In a nutshell, even in taking Bohn’s (1998) date conventions and definition of the fiscal reaction function, assuming only that $\lim_{k \to +\infty} E_t \left[ u_{t,k} \cdot \prod_{j=1}^{k} (1 + y_{t+j}) \right] = 0$ and that $\mu_t$ is a weakly stationary process does not imply that $\lim_{k \to +\infty} E_t \left[ u_{t,k} \cdot \prod_{j=1}^{k} (1 + y_{t+j}) \cdot \mu_{t+k} \right] = 0$, which is essential for the proof to work.

A.2.2 An example with strong stationarity

Let $u_k \sim iid U([0,1])$ and $\mu_k = -\ln(u_k)$, defined almost everywhere with mean and variance 1. Since its mean and variance exist, the strongly stationary $\mu_k$ is also weakly stationary.

Let $X_k \equiv \frac{1}{k^2 \ln(\ln(u_k) - 1)^2}$. Then, using a generalized integral:

$$
E[X_k] = \frac{1}{k^2} \int_{0}^{1} \frac{du}{u(\ln u - 1)^2} = \frac{1}{k^2} \left[ \frac{-1}{\ln u - 1} \right]_{0}^{1} = \frac{1}{k^2} \to 0
$$

But $\forall k \in N^*$, $E[X_k \mu_k] = +\infty$. Indeed for $0 < a < 1$, we have:

$$
\int_{a}^{1} \frac{-\ln u \, du}{u(\ln u - 1)^2} = \left[ \frac{\ln u}{\ln u - 1} \right]_{a}^{1} - \int_{a}^{1} \frac{du}{u(\ln u - 1)} = \left[ -\frac{\ln u}{\ln u - 1} \right]_{a}^{1} - \ln(1 - \ln u)_{a}^{1} \to +\infty
$$
With these $\mu_k$ and $u_{t,k} \prod_{j=1}^{k} (1 + y_{t+j}) = X_k$ processes, the transversality condition is therefore violated. Of course this example remains valid with both Bolin’s and our notations for debt and surplus ratios.

A.3 Debt cannot be at the same time $I(m)_{m \geq 0}$ in level and $I(k)_{k \geq 0}$ after discounting

If $X_t \sim I(m)$ and $X_0$ is known, then $\mathbb{V}[X_t] \sim O(t^{2m})$. This result can be shown by induction, the assertion being trivial for $m = 0$. Let us assume the result established for every $k < m$, and let us consider a process $X_t \sim I(m)$. The process $Y_t = (1-L)X_t$ can then be defined and it is $I(m-1)$. The variance of $X_t = X_0 + Y_t + ... + Y_1$ equals $\sum_{j=1}^{t} \mathbb{V}[Y_j] + \sum_{j_1, j_2 = 1, j_1 \neq j_2}^{t} \mathbb{Cov}[Y_{j_1}, Y_{j_2}]$. The first term is a sum over $1 \leq j \leq t$ of $O(j^{2m-2})$ terms thanks to the induction hypothesis, so it is $O(t^{2m-1})$. The second term, with Cauchy-Schwarz inequality, is a sum on $1 \leq j_1 \neq j_2 \leq t$ of $O(j_1^{m-1}j_2^{m-1})$ terms, thus is $O(t^{2m})$. Eventually we did prove that the variance of the process $X_t$ is $O(t^{2m})$.

Furthermore, with $X_t \sim I(m)$ and $Y_t \sim I(m')$ for $m' \leq m$, Cauchy-Schwarz inequality allows to write $\mathbb{Cov}[X_t, Y_t] = O(t^{2m})$. One can establish by induction on $k \geq 0$ that there exists coefficients $\{\alpha_{k,j}\}_{0 \leq j \leq k}$ such that:

$$\Delta^k \left[ \frac{D_{t+n}}{(1+r)^n} \right] = \frac{1}{(1+r)^n} \sum_{j=0}^{k} \alpha_{k,j} \Delta^j D_{t+n}$$

The $\{\alpha_{k,j}\}_{0 \leq j \leq k}$ can be obtained by the following recursion: $\alpha_{k+1,j} = (1+r)\alpha_{k,j-1} - r\alpha_{k,j}$ with $\alpha_{0,0} = 1$, $\alpha_{k,j} = 0$ for $j < 0$ and $j > k$.

Developing for $k \geq 0$:

$$\mathbb{V} \left[ \Delta^k \left[ \frac{D_{t+n}}{(1+r)^n} \right] \right] = \frac{1}{(1+r)^{2n}} \mathbb{V} \left[ \sum_{j=0}^{k} \alpha_{k,j} \Delta^j D_{t+n} \right]$$

And we notice that the term whose variance we look at is a sum of $k+1$ terms all integrated of order smaller or equal than $m$. Developing the variance enables to write it as a sum of $(k+1) + k(k+1)$ terms which are all $O((t+n)^{2m})$. We can then infer that $\mathbb{V} \left[ \Delta^k \left[ \frac{D_{t+n}}{(1+r)^n} \right] \right] = O((t+n)^{2m}/(1+r)^{2n})$, which means that for every $k \geq 0$, this variance will converge towards 0 when $n$ goes to infinity. Hence, the $k$ times differentiated discounted debt cannot be a stationary non-zero process for any $k \geq 0$.

A.4 Proof of proposition I.1

**Proposition I.1.** Let $f$ be a deterministic and discrete function of time.

1. If $D_{t+n}/f(n) \sim I(m)$, with $m \geq 0$ and $f(n) = o((1+r)^n/n^m)$, then debt verifies the transversality condition (TC ad hoc).
2. If $D_{t+n}/f(n) \sim I(0)$ with $f(n) = O((1+r)^n)$ and $\mathbb{E}[D_{t+n}/f(n)] = 0$, then debt verifies the transversality condition (TC ad hoc).
The proof of the first part is very close to that of proposition 1 in Bohn (2007). Noting \( d_{t+n} = D_{t+n}/f(n) \) and assuming that this process is \( I(m) \), we can write, with Bohn’s notations:

\[
d_{t+n} = \sum_{k=0}^{m-1} p_k(n) \Delta^k d_t + \sum_{i=1}^{n} p_{m-1}(i) \Delta^m d_{t+(n+1-i)}
\]

This implies:

\[
E_t[D_{t+n}]/(1+r)^n = \frac{f(n)}{(1+r)^n} \left( \sum_{k=0}^{m-1} p_k(n) \Delta^k d_t + \sum_{i=1}^{n} p_{m-1}(i) E_t[\Delta^m d_{t+(n+1-i)}] \right).
\]

Since \( \Delta^k d_t \) is known at time \( t \) and \( p_k(n) = O(n^k) \), \( \sum_{k=0}^{m-1} p_k(n) \Delta^k d_t = O(n^{m-1}) \).

Moreover, \( E_t[\Delta^m d_{t+(n+1-i)}] \xrightarrow[n \to +\infty]{} \mu \equiv E[\Delta^m d_t] \) and \( \sum_{i=1}^{n} p_{m-1}(i) = O(n^m) \), implying \( \sum_{i=1}^{n} p_{m-1}(i) E_t[\Delta^m d_{t+(n+1-i)}] = O_p(n^m) \).

Finally, \( \left( \sum_{k=0}^{m-1} p_k(n) \Delta^k d_t + \sum_{i=1}^{n} p_{m-1}(i) E_t[\Delta^m d_{t+(n+1-i)}] \right) = O_p(n^m) \).

Hence, \( E_t[D_{t+n}]/(1+r)^n \xrightarrow[n \to +\infty]{} 0 \).

The proof of the second part is straightforward noting that \( \lim_{n \to +\infty} E_t[D_{t+n}/f(n)] = E[D_{t+n}/f(n)] = 0 \) given the stationarity of \( D_{t+n}/f(n) \).

### A.5 Description of the French quarterly data

Quarterly data on general government net surplus \( S_t \) (B9A) as well as interest payments \(^{21}\) are extracted from the quarterly national accounts (base 2000, INSEE). General government quarterly primary surplus can be simply deduced from those two series. The corresponding cyclically-adjusted primary surplus requires the computation of an output gap at the same frequency. We therefore construct a quarterly potential GDP with a quadratic interpolation of the annual potential GDP provided by the European Commission (in base 2005, not available in base 2000). We can then compute a quarterly output gap for France using real quarterly GDP (quarterly national accounts in base 2005 as well). Using the same elasticity of primary surplus with respect to output gap as for annual data (0.5), we obtain a quarterly cyclically-adjusted primary surplus.

General government net financial debt is only published on an annual basis. Its evolution is linked to surplus by the following equation:

\[
\Delta D_{net,t} = -S_t + Adj_t
\]

where \( Adj_t \) is an adjustment term reflecting revaluation of assets and liabilities in the government balance sheet.

The quarterly evolution of net financial debt can be deduced from the quarterly surplus. Chow and Lin’s (1971) method is used to compute the adjustment term at a quarterly frequency under the constraint that debt annual evolution is equal

\(^{21}\) We use net interest paid (D41 paid minus received) to be coherent with of our choice of the net debt ratio.
to its 4 quarterly evolutions over the year. Surplus and net financial debt at a quarterly frequency are finally divided by nominal quarterly GDP (same date for debt, previous date for surplus) in order to estimate the fiscal reaction function at a quarterly frequency. Note that the coefficient on debt is therefore divided by about 4.

References


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